NONLINEAR LANGMUIR WAVES IN ULTRA-RELATIVISTIC DEGENERATE QUANTUM ELECTRON-ION PLASMAS

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Abstract

The nonlinear Langmuir waves (NLWs) in a fully degenerate ultra-relativistic quantum plasma are considered. The equations of state for ultra-relativistic hot electrons, with quantum hydrodynamic (QHD) model, are used to drive a non-linear differential equation for the chemical potential describing these waves in the plasma system. It is found that the ordinary electronic oscillations, similar to the classical oscillations, occur along with small-scale quantum Langmuir oscillations induced by the Bohm quantum force. These investigations can be helpful in understanding the nonlinear features of NLWs in various astrophysical plasmas such as planetary interiors and compact astrophysical objects.

Keywords: *Chemical potential, Electron-ion plasmas, Quantum ultra-relativistic degenerate plasma.*

INTRODUCTION

Over the last few years, the theory of degenerate quantum plasma became one of the most interesting areas in plasma physics. There are many researches and achievements associated with the theory of collective processes in this type of plasma presented in reviews [1-4]. The collective processes occurring in quantum plasma are extensively studied by several hundred of researchers who are describing just ion-acoustic waves (IAWs) in such plasma. However, till now, a very few papers on the theory of Langmuir waves in quantum plasma are established [5-13] some of them are restricted to the dispersion of Langmuir waves (LWs) in the linear approximation [4,5].

The propagation of LWs in plasma with warm ($T\div 0$) quantum-depleted electrons is studied based on a gas-dynamic approach [2,11,14,15], assuming that the electron gas obeys the ultra-relativistic equation of state. A nonlinear differential equation describing oscillations of the chemical potential is obtained and numerically solved. Dubinov and Kitayev [12,13] investigated the presence of dichromatic LWs in fully degenerate quantum plasma. They showed that, the low-frequency component of the waves corresponds to classical LWs, while the high-frequency component is due to free-electron quantum oscillations. The model of Dubinov and Kitayev [12,13] treated the non-relativistic degenerate quantum plasma. However, the current work is an attempt to apply gas-dynamic approach [14] to ultra-relativistic quantum-degenerated electrons in ion-electron warm plasma system.

In this article, we derived a new non-linear differential equation describing the isothermal Langmuir waves for the chemical potential in ultra-relativistic quantum plasma [16]. The quantum principles are taken into consideration. So, we include the quantum Bohm force into the original gas-dynamic equations and we used the equation of state for electrons in warm electronic Fermi gas to drive the number density of ultra-relativistic electrons in terms of the chemical potential. The effects of quantum diffraction parameter, the phase velocity, and the density of electrons (equivalently the Fermi energy or the Fermi temperature) on chemical potential are addressed. In Sec. II, we used the equation of state for a Fermi gas of warm degenerate electrons to drive the number density of electrons. In Sec III we present the basic equations of our theoretical model and derived the non-linear differential equation describing the isothermal Langmuir waves. Numerical results and conclusions are given in Sec. IV.

EQUATION OF STATE FOR DEGENERATE ELECTRON FERMI GAS

In this paper we consider an ultra-relativistic high density plasma system, consists of quantum-degenerated electrons and immobile ions. By degenerate we mean that the electrons obey the Fermi-Dirac statistics. Examples of such kind of plasma is detected in space (neutron stars, white dwarfs, etc. [17-19] and in laboratory experiments such as semiconductors and metals [20].

For ultra-relativistic high density electrons, the kinetic energy is given by

$$E = \sqrt{P^2c^2 - m_e^2c^4}, \tag{1}$$

where *c* is the light velocity, *P* and m_e are the momentum and mass of the moving electron, respectively. From quantum mechanical point of view the electrons have 1/2 - integer spin, *s*, so there is always a degeneracy factor, $g_s = (2s+1)$, of states.
Hence, the density of states of relativistic electronic Fermi gas with kinetic energy, *E*, moving in a volume, *V*, is given by[14]

$$G(E) = \frac{g_s V \quad E}{2\pi^2 \hbar^3 c^3} \sqrt{E^2 - m_e^2 c^4}. \qquad (2)$$

For the ultra-relativistic electrons, $E >> m_e c^2$, and hence the density of states is expanded as

$$G(E) = \frac{V}{\pi^2 \hbar^3 c^3} \left(E^2 - \frac{1}{2} m_e^2 c^4 + ... \right), \qquad (3)$$

According to statistical mechanics, the number of electrons of Fermi gas can be expressed as [14]

$$N_e\left(\mu, T\right) = \int_0^\infty \frac{G(E) dE}{Z^{-1} e^{\beta E} + 1}. \qquad (4)$$

Here, $Z = e^{\beta\mu}$ and $\beta = \frac{1}{k_B T_F}$, where k_B is the Boltzmann constant, T_F is the Fermi electron gas temperature and μ is the chemical potential of the electrons. It is defined as the value of Fermi energy at $T = 0$. Substituting *G*(*E*) from Eq.(3) into Eq.(4) gives

$$n_e\left(\mu, T\right) = \frac{N_e\left(\mu, T\right)}{V} = \frac{1}{\pi^2 \hbar^3 c^3} \int_0^\infty \frac{\left(E^2 - \frac{1}{2} m_e^2 c^4 + ... \right) dE}{Z^{-1} e^{\beta E} + 1}. \qquad (5)$$

The integral in Eq.(5) can be performed analytically with Sommerfeld expansion to give

$$n_e\left(\mu, T\right) = \frac{8\pi}{\left(hc \right)^3} \left(\frac{\mu^3}{3} + \left(\frac{\pi^2}{3\beta^2} - \frac{1}{2} m_e^2 c^4 \right) \mu + ... \right). \qquad (6)$$

For the ultra-relativistic high density electrons the pressure is related to the electron density by the relation

$$p_e\left(\mu, T\right) = \frac{2}{3} E_k = \frac{2}{3} \frac{\hbar c}{4\pi^2} [3\pi^2 n_e\left(\mu, T\right)]^{\frac{4}{3}}, \qquad (7)$$

where E_k is the relativistic kinetic energy of electrons.

The dynamics describing the fluid electronic gas motion in the mentioned degenerate quantum plasma can be expressed by the continuity equation;

$$\frac{\partial n_e}{\partial t} + \frac{\partial \left(n_e v_e \right)}{\partial x} = 0, \qquad (8)$$

the momentum balance equation;

$$\frac{\partial v_e}{\partial t} + v_e \frac{\partial v_e}{\partial x} = \frac{e}{m_e}\frac{\partial \phi}{\partial x} - \frac{1}{m_e n_e}\frac{\partial p}{\partial x} + \frac{\hbar^2}{2m_e^2}\frac{\partial}{\partial x}\left(\frac{\partial^2 \sqrt{n_e} \,/\, \partial x^2}{\sqrt{n_e}}\right), \quad (9)$$

and Poisson's equation for the self-consistent potential, ϕ

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e\left(n_e - n_i\right). \quad (10)$$

where n_e is given by Eq.(6).

In Eqs. (6)-(10) m_e, n_e and v_e represent, respectively, the mass, density and fluid velocity for electrons, n_i is density of ions and $\hbar$ stands for the scaled Planck's constant. The term proportional to $\hbar^2$ is the quantum force due to the so-called Bohm potential[21]. In Eq.(6), the higher order terms in μ can be ignored since their contributions to $n_e\left(\mu, T_F\right)$ are negligible.

Eqs. (6)-(10) can be rewritten in a dimensionless form by the following normalizations

$$X = \frac{\omega_e}{C_s}x,\ \ T = t \ \ \omega_e,\ \ N_e = \frac{n_e}{n_{eo}},\ \ V_e = \frac{v_e}{C_s},\ \ \psi = \frac{e\phi}{E_F},\ \ \bar{\mu} = \frac{\mu}{E_F},\ \ P_e = \frac{p_e}{E_k},$$

where $n_{e0} \simeq \frac{8\pi E_F^3}{\left(\hbar c\right)^3} c_1$, and

$$c_1 = \frac{\mu_0^3}{3} + \left(\frac{\pi^2}{3\beta^2} - \frac{1}{2}m_e^2 c^4\right)\frac{\mu_0}{E_F^2}, \quad (11)$$

$E_F = \left(3\pi^2 n_{eo}\right)^{\frac{1}{3}} \hbar c,\ \omega_e = \sqrt{\frac{4\pi e^2 n_{e0}}{m_e}}$ and $C_s = \sqrt{\frac{E_F}{m_e}}$. Hence, Eqs.(6)-(10) become

$$\frac{\partial N_e}{\partial T} + \frac{\partial\left(N_e V_e\right)}{\partial X} = 0, \quad (12)$$

$$\frac{\partial V_e}{\partial T} + V_e \frac{\partial V_e}{\partial X} = \frac{\partial \psi}{\partial X} - \frac{2}{3} \quad N_e^{-\frac{2}{3}} \frac{\partial N_e}{\partial x} + \frac{H^2}{2}\frac{\partial}{\partial X}\left(\frac{\frac{\partial^2 \sqrt{N_e}}{\partial X^2}}{\sqrt{N_e}}\right), \quad (13)$$

$$\frac{\partial^2 \psi}{\partial X^2} = N_e + 1, \quad (14)$$

$$N_e = \frac{1}{c_1}\left(\frac{\overline{\mu}^3}{3} + \left(\frac{\pi^2}{3\beta^2} - \frac{1}{2}m_e^2c^4\right)\frac{\overline{\mu}}{E_F^2}\right) = \frac{1}{c_1}\left(\frac{\overline{\mu}^3}{3} + c_2\overline{\mu}\right), \qquad (15)$$

where $H^2 = \frac{\hbar^2}{m_e^2}\frac{\omega_{ec}^2}{C_s^4}$ and $c_2 = \frac{1}{E_F^2}\left(\frac{\pi^2}{3\beta^2} - \frac{1}{2}m_e^2c^4\right)$. (16)

THE EVOLUTION EQUATION FOR $\mu(\xi)$

The solution of Eqs. (12)-(14), can be developed by assuming that the solution is stationary wave, moving with a phase velocity *U* and the dependent variables, $N_e, V_e,\ \psi$, and $\overline{\mu}$ depend on the new coordinate $\xi = X - UT$. Hence, Eqs. (12)-(14) are transformed into the following system of ordinary differential equations

$$-U\frac{dN_e}{d\xi} + \frac{d}{d\xi}(N_eV_e) = 0, \qquad (17)$$

$$-U\frac{dV_e}{d\xi} + V_e\frac{dV_e}{d\xi} = \frac{d\psi}{d\xi} - \frac{2}{3}\quad N_e^{-\frac{2}{3}}\frac{\partial N_e}{\partial \xi} + \frac{H^2}{2}\frac{d}{d\xi}\left(\frac{1}{N_e}\left[\frac{d^2N_e}{d\xi^2} - \frac{1}{2N_e}\left(\frac{dN_e}{d\xi}\right)^2\right]\right) = 0, \qquad (18)$$

$$\frac{d^2\psi}{d\xi^2} = N_e + 1, \qquad (19)$$

Integrating Eqs. (17) and (18) under the conditions; $N_e = n_{e0}, \psi = \psi_0,\ \mu = \mu_0$ and $V_{e0} = 0,$ are fulfilled for the unperturbed quasineutral plasma gives

$$\psi - \psi_0 = -\frac{U^2}{2}\left[1 - \left(\frac{1}{N_e}\right)^2\right] + 2\quad N_e^{\frac{1}{3}} - \frac{H^2}{2}\frac{1}{N_e}\left[\frac{d^2N_e}{d\xi^2} - \frac{1}{2N_e}\left(\frac{dN_e}{d\xi}\right)^2\right]. \qquad (20)$$

Based on Eq. (15), the differentiations $\frac{dN_e}{d\xi}$ and $\frac{d^2N_e}{d\xi^2}$ can be substituted in Eq. (20) to give

$$\psi - \psi_0 = f_1(\mu) + f_2(\mu)\frac{d^2\mu}{d\xi^2} + f_3(\mu)\left(\frac{d\mu}{d\xi}\right)^2, \qquad (21)$$

where

$$f_1(\mu) = -\frac{U^2}{2}\left[1-\left(\frac{c_1}{\left(\frac{\overline{\mu}^3}{3}+c_2\overline{\mu}\right)}\right)^2\right]+\frac{2}{c_1}\left(\frac{\overline{\mu}^3}{3}+c_2\overline{\mu}\right)^{1/3}, \tag{22}$$

$$f_2(\mu) = -\frac{H^2}{2}\frac{\left(\overline{\mu}^2+c_2\right)}{\left(\frac{\overline{\mu}^3}{3}+c_2\overline{\mu}\right)}, \tag{23}$$

and $$f_3(\mu) = -\frac{H^2}{2}\frac{1}{\left(\frac{\overline{\mu}^3}{3}+c_2\overline{\mu}\right)}\left[2\overline{\mu}-\frac{\left(\overline{\mu}^2+c_2\right)^2}{2\left(\frac{\overline{\mu}^3}{3}+c_2\overline{\mu}\right)}\right]. \tag{24}$$

Differentiate Eq.(21) twice with respect to ξ and substitute $\frac{d^2\psi}{d\xi^2}$ from Eq.(19) and N_e from Eq. (15), in the resulting equation one can get

$$\begin{gathered} g_1(\mu)\frac{d^4\mu}{d\xi^4}+g_2(\mu)\frac{d\mu}{d\xi}\frac{d^3\mu}{d\xi^3}+g_3(\mu)\left(\frac{d^2\mu}{d\xi^2}\right)^2+g_4(\mu)\left(\frac{d\mu}{d\xi}\right)^2\frac{d^2\mu}{d\xi^2}+g_5(\mu)\frac{d^2\mu}{d\xi^2} \\ +g_6(\mu)\left(\frac{d\mu}{d\xi}\right)^4+g_7(\mu)\left(\frac{d\mu}{d\xi}\right)^2 = g_0(\mu), \end{gathered} \tag{25}$$

where

$$g_0(\mu) = \frac{1}{c_1}\left(\frac{\overline{\mu}^3}{3}+c_2\overline{\mu}\right)+1, \qquad g_1(\mu) = f_2(\mu) = -\frac{H^2}{2}\frac{\left(\overline{\mu}^2+c_2\right)}{\left(\frac{\overline{\mu}^3}{3}+c_2\overline{\mu}\right)},$$

$$g_2(\mu) = 2f_3(\mu)+2\frac{df_2(\mu)}{d\mu} = \frac{3H^2\left(\mu^2-3c_2\right)^2}{2\left(3c_2\mu+\mu^3\right)^2},$$

$$g_3(\mu) = 2f_3(\mu)+\frac{df_2(\mu)}{d\mu} = \frac{9c_2H^2\left(c_2-\mu^2\right)}{\left(3c_2\mu+\mu^3\right)^2},$$

$$\begin{aligned} g_4(\mu) &= 5\frac{df_3(\mu)}{d\mu}+\frac{d^2f_2(\mu)}{d\mu^2} \\ &= \frac{9H^2\left(c_2\left(17\mu^4-21c_2\left(c_2+\mu^2\right)\right)+\mu^6\right)}{2\left(3c_2\mu+\mu^3\right)^3}, \end{aligned}$$

$$g_5(\mu) = \frac{df_1(\mu)}{d\mu} = \frac{\left(c_2+\mu^2\right)}{3c_1}\left(\frac{2}{\left(c_2\mu+\frac{\mu^3}{3}\right)^{2/3}}-\frac{81c_1^3U^2}{\left(3c_2\mu+\mu^3\right)^3}\right),$$

$$g_6(\mu) = \frac{d^2 f_3(\mu)}{d\mu^2} = -\frac{9H^2\left(\mu^2 - 3c_2\right)\left(3c_2\left(5c_2\mu^2 + 3c_2^2 + 5\mu^4\right) + \mu^6\right)}{2\left(3c_2\mu + \mu^3\right)^4},$$

$$g_7(\mu) = \frac{d^2 f_1(\mu)}{d\mu^2} = \frac{27c_1^2U^2\left(12c_2\mu^2 + 9c_2^2 + 7\mu^4\right)}{\left(3c_2\mu + \mu^3\right)^4} + \frac{4c_2\left(\mu^2 - c_2\right)}{\sqrt[3]{3}c_1\left(3c_2\mu + \mu^3\right)^{5/3}}.$$

Eq.(25) is an autonomous fourth-order differential equation for the chemical potential, $\mu(\xi)$.

NUMERICAL RESULTS AND DISCUSSION

In this section we investigated the effects of different plasma parameters for our model on the behavior of chemical potential, μ. Namely the variations of the electron number density, n_{e0}, (or equivalently, E_F or T_F where n_{e0}, E_F and T_F are related according to; $E_F = K_B T_F = \left(3\pi^2 n_{eo}\right)^{\frac{1}{3}} \hbar c)$, quantum diffraction parameter, *H* and the phase velocity, *U*, on the properties of LWs are studied. For this purpose, Eq.(25) is solved by one-step numerical methods such as Runge-Kutta algorithm.

At first we address the case of negilgable quantum effects. In this case the quantum diffraction parameter, *H* is setting equal to zero and hence, Eq.(25) is reduced to the following autonomous second order differential equation:

$$g_5(\mu)\frac{d^2\mu}{d\xi^2} + g_7(\mu)\left(\frac{d\mu}{d\xi}\right)^2 = g_0(\mu). \qquad (26)$$

By solving this equation numerically by one-step Runge-Kutta algorithm with the initial conditions; $\mu(0) = 2,$ and $\left(\frac{d\mu}{d\xi}\right)_{\xi=0} = 0.01$, one can obtain variable profiles of $\mu(\xi)$ for a LW in ultra-relativistic non quantum plasma. Figure 1 illustrates that the increase in the electron number density, n_{e0}, leads to a decrease in the chemical potential, $\mu(\xi)$. On the other hand, the increase in the phase velocity, *U*, leads to an increase in the chemical potential, $\mu(\xi)$, as shown in Fig.(2). In both figures $\mu(\xi)$ is decreasing with ξ.

In the second case we take the quantum effects into account. Sitting the initial conditions; $\mu(0) = 2,$ $\left(\frac{d\mu}{d\xi}\right)_{\xi=0} = 0.01$ and $\left(\frac{d^2\mu}{d\xi^2}\right)_{\xi=0} = \left(\frac{d^3\mu}{d\xi^3}\right)_{\xi=0} = 0$, one can solve Eq.(25) numerically to obtain variable profile of $\mu(\xi)$ for a LW in ultra relativistic quantum plasma. Figures 3-5 show oscillatory decreasing variation of $\mu(\xi)$ with ξ at different values of quantum diffraction parameter, *H*, released from the Bohm potential effect. From these figures, one can see that the resulting wave is a superposition of two oscillating waves. The oscillation frequency is decreased with the increasing of *H*. Further, the small scale wave oscillations appear due to the quantum Bohm force, Eq.(25) while the large scale wave oscillations results when the quantum Bohm force excluded. The small-scale oscillations are called quantum Langmuir oscillations[12]. It is noted that solutions featuring similar quantum oscillations were discovered in a numerical study of non-linear ion-acoustic waves in a quantum-ion plasma and in a study of non-linear electron-acoustic waves in a quantum electron-ion plasma with two-temperature electrons[22,23].

General concluding, a fourth order non-linear ordinary differential equation is derived to describe the LWs in an ultra-relativistic warm quantum electron-ion plasma. Numerical results show that the profile of the chemical potential reveals small scale quantum Langmuir oscillations attributed to the quantum Bohm force [21].

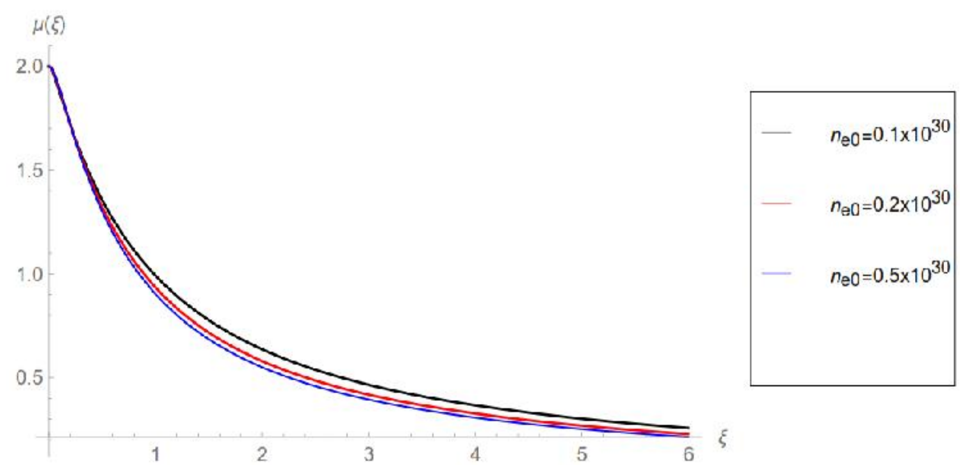

Figure 1: (Color online) Profile of the chemical potential, $\mu(\xi)$, for electrons of a LW in ultra-relativistic non quantum plasma at different values of the unperturbed electron density, n_{e0}, with chemical potentials, ($\mu_0 = 2$ and $U = 0.5$).

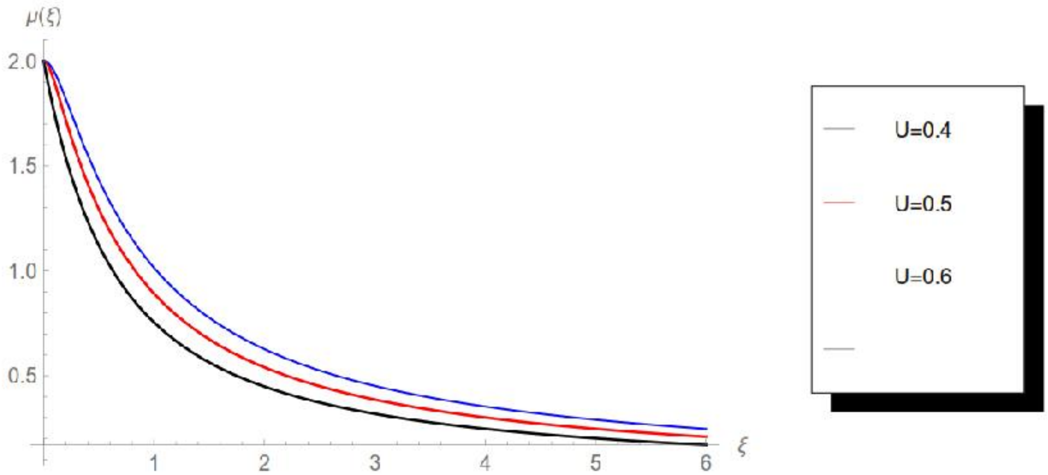

Figure 2: (Color online) Profiles of the chemical potential, $\mu(\xi),$ for electrons of a LW in ultra-relativistic non quantum plasma at different values of phase velocity, U ($\mu_0 = 2$ and n_{e0}).

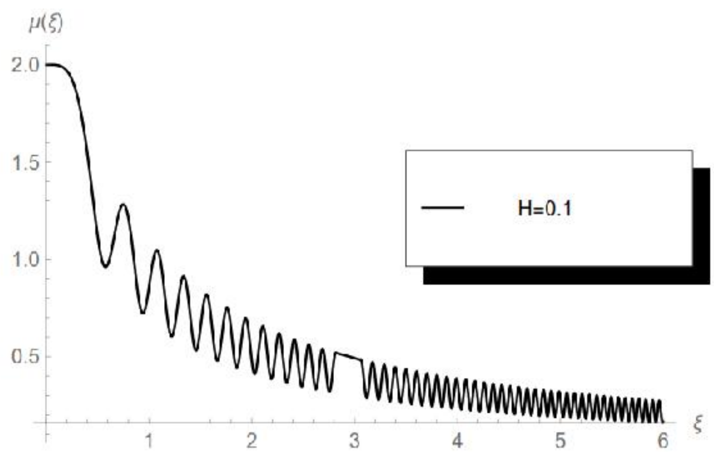

Figure 3: The profile of the chemical potential, $\mu(\xi),$ for electrons of a LW in ultra-relativistic quantum plasma at $H = 0.1(U = 0.5\ \ \mu_0 = 2$ and $n_{e0} = 10^{30}$).

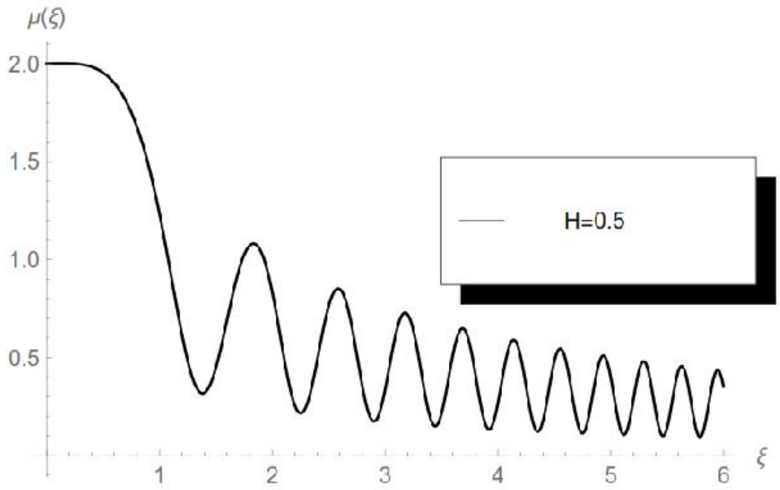

Figure 4: The profile of the chemical potential, $\mu(\xi)$, for electrons of a LW in ultra-relativistic quantum plasma at $H = 0.5$ ($U = 0.5$ $\mu_0 = 2$ and $n_{e0} = 10^{30}$).

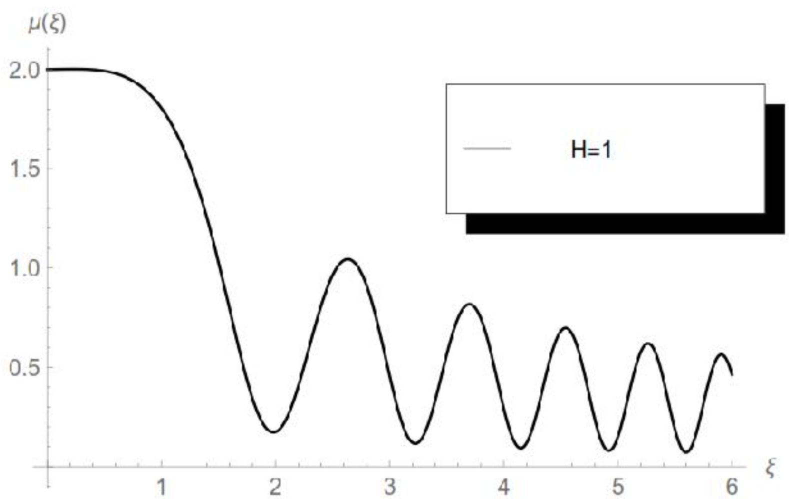

Figure 5: The profile of the chemical potential, $\mu(\xi)$, for electrons of a LW in ultra-relativistic quantum plasma at $H = 1$ ($U = 0.5$ $\mu_0 = 2$ and $n_{e0} = 10^{30}$).

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